A Cautionary Note on Pricing Longevity Index Swaps
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Objectives

- Pricing QxX index swap
- Examining the parameter risk and model risk in the pricing
- Determining the effect of the uncertainty on the pricing

Outline

- Mortality derivatives
- QxX index Swap
- Parameter risk
- Model risk
- Conclusion

Mortality Derivatives

What are mortality derivatives?

- Financial contracts that have payoffs tied to the level of a certain longevity or mortality index
- Examples: survivor bond, survivor swap, ...

How to price mortality derivatives?

- Mortality model
- Wang's Transform, Q measure, ...

A two-factor stochastic mortality model (Cairns, Blake and Dowd (2006))

Mathematical Specification:

$$\ln \frac{q_{x,t}}{1 - q_{x,t}} = A_1(t) + A_2(t)x. \tag{1}$$

- $x \rightarrow$ age
- $t \rightarrow$ time
- $q_{x,t} \rightarrow$ realized single-year death probability
- $\{A_1(t)\}$ and $\{A_2(t)\} \rightarrow$ discrete-time stochastic processes

A two-factor stochastic mortality model(con't)

Stochastic Mortality: Recall: $\ln \frac{q_{x,t}}{1-q_{x,t}} = A_1(t) + A_2(t)x$

$$\begin{aligned} D(t+1) &= A(t+1) - A(t) \\ &= \mu + CZ(t+1) \end{aligned} \tag{2}$$

- $A(t) = (A_1(t), A_2(t))'$
- μ →constant 2×1 vector
- C →constant 2×2 upper triangular matrix
- $Z(t)$ →2-dim standard normal random variable

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Model fitting

Data

- $q_{x,t}, \quad x = 65, 66, \ldots, 109, \quad t = 1971, 1972, \ldots, 2005$

Model fitting

$$\ln \frac{q_{x,t}}{1-q_{x,t}} = A_1(t) + A_2(t)x \qquad D(t+1) = \mu + CZ(t+1)$$

- First step: Estimate $A(t)$ by least square method
- Second step: Estimate μ and C through maximum likelihood estimation

Forecasting

Steps

$$\ln \frac{q_{x,t}}{1-q_{x,t}} = A_1(t) + A_2(t)x \qquad D(t+1) = \mu + CZ(t+1)$$

- Simulate a set of Z
- Obtain corresponding $D(2005+k), \quad k = 1, 2, \ldots, 10$
- $A(2005+k) = A(2005) + \sum_{n=1}^{k} D(2005+n), \quad k = 1, 2, \ldots, 10$
- Calculate $q_{x,2005+k}$

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Pricing in Risk-adjusted world

Real-world probability measure(P measure)

$$D(t+1) = \mu + CZ(t+1) \quad (3)$$

Risk-adjusted probability measure(Q measure)

$$\begin{aligned} D(t+1) &= \mu + C(\tilde{Z}(t+1) - \lambda) \\ &= \tilde{\mu} + C\tilde{Z}(t+1), \end{aligned} \quad (4)$$

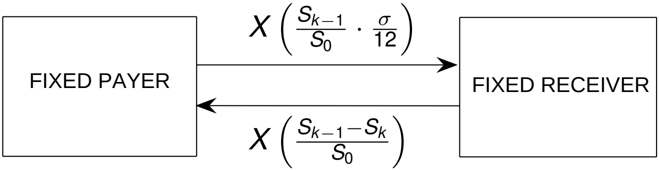
where λ is the market price of risk and $\tilde{\mu} = \mu - C\lambda$.

QxX Index

"allows market participants to measure, manage and trade exposure to longevity and mortality risks in a standardized, transparent, and real-time manner"

- Launched by Goldman Sacs in 2007
- Based on a reference pool consisting of a set of lives underwritten by AVS Underwriting LLC
- The index value is the number of lives in the reference pool
- Published monthly, providing "real-time" mortality information

Payment structure of QxX index swap



- X →nominal amount
- S_k →index value in the kth month
- σ →fixed spread
- Goldman Sacs: $\sigma = 500$ basis points for 10-year swap

10-year QxX index swap price

- QxX index swap is priced by determining the “fair" spread σ

$$\begin{array}{rl} & \text{Market value of future payments from fixed payer} \\ = & \text{Market value of future payments from fixed receiver} \end{array}$$

- We need to know the market price of risk λ. In our analysis,
  - Not enough data to estimate λ for QxX index swaps
  - Use the estimated market price of risk from BNP/EIB longevity bond

10-year QxX index swap price (Con't)

Estimates of σ (in basis points) under different choices of $\lambda = (\lambda_1, \lambda_2)$

λ_1	λ_2	σ
0.375	0	627
0	0.316	619
0.175	0.175	622

Why $\sigma \neq 500$ bps?

- No access to the actual QxX index reference pool
- Lack of market data for the swap
- Existence of parameter risk and model risk

Parameter risk under Bayesian Method

- $D(t) \sim \mathrm{MVN}(\mu, V)$, where $V = C'C$.
- Treat μ and C as random variables

$$D(t) \mid \mu, V \sim \mathrm{MVN}(\mu, V) \tag{5}$$

- Use a non-informative prior distribution

$$\pi(\mu, V) \propto |V|^{-3/2} \tag{6}$$

- Marginal posterior distribution

$$\begin{aligned} V^{-1} \mid D &\sim \mathrm{Wishart}(n-1, n^{-1}\hat{V}^{-1}), \\ \mu \mid D &\sim \mathrm{MVN}(\hat{\mu}, n^{-1}\hat{V}), \end{aligned} \tag{7}$$

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Estimated marginal posterior density functions for the model parameters

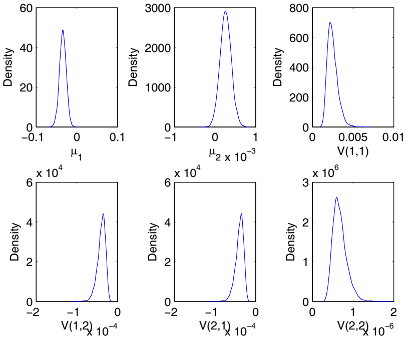

Figure: Simulated marginal posterior parameter distributions. (We denote the ith element in μ by μ_i and the (j,k)th element in V by $V_{j,k}$).

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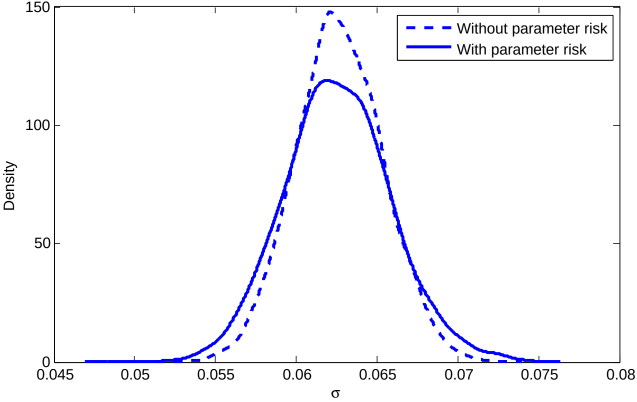
Simulated predictive distribution of σ, $\lambda = (0.375\,,0)$



95% Confidence Interval for σ

λ_1	λ_2	With parameter risk	Without parameter risk
0.375	0	(560,693)	(574,680)
0	0.316	(553,685)	(567,673)
0.175	0.175	(557,686)	(571,675)

Table: 95% confidence intervals for σ (in basis points) under different choices of λ_1 and λ_2.

Model risk in pricing

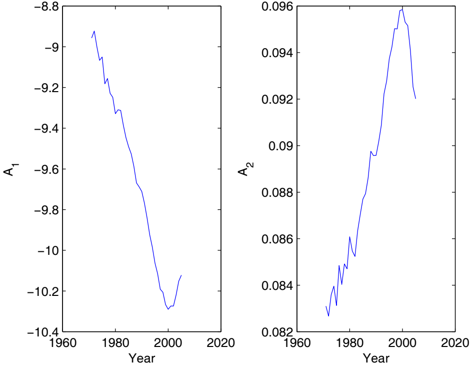
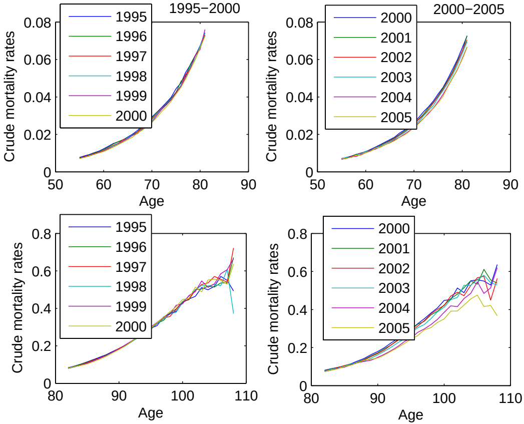


Figure: Estimated values of $A_1(t)$ and $A_2(t)$, 1971–2005.

What causes the reverse trend?

Crude mortality curves

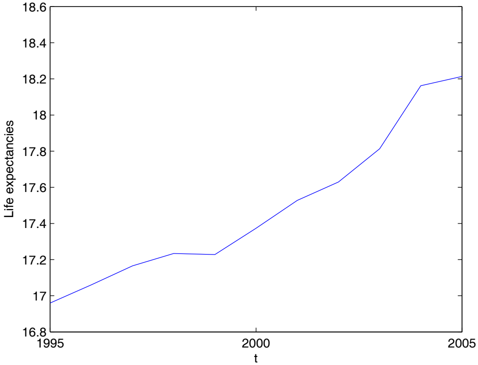
$$\ln \frac{q_{x,t}}{1-q_{x,t}} = A_1(t) + A_2(t)x$$



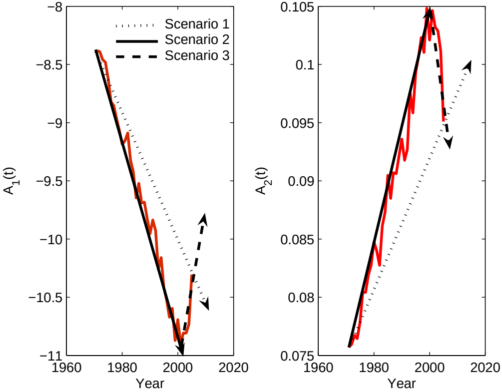
What causes the reverse trend?

Life expectancies at age 65

$$\ln \frac{q_{x,t}}{1-q_{x,t}} = A_1(t) + A_2(t)x$$



Three possible scenarios



How does the change affect QxX index swap price?

λ_1	λ_2	Swap spread, σ Scenario 1	Scenario 2	Scenario 3
0.375	0	627	674	566
0	0.316	619	683	553
0.175	0.175	622	678	558

Table: Swap spread (in basis points) under three different scenarios.

Conclusion

- The swap spread computed from our pricing framework is fairly close to the spread currently offered by Goldman Sachs
- The pricing is still very experimental
  - Parameter risk and model risk are significant in the pricing
  - No sufficient market price data to estimate market prices of risk
  - No clear conclusion on how mortality rates may evolve in the future

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