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# Applying Fuzzy Optimization to Risk Assessment

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**ABSTRACT**
Risk assessment and management involves identifying, selecting, and implementing measures that can be applied to mitigate risk in a particular situation [Better et al., 2008]. Many qualitative and quantitative techniques have been developed to help risk managers achieve this objective. Optimization is an example of such a quantitative technique.

The process of optimization aims at finding the optimal solution to a given problem. Generally the solution must satisfy some constraints and objective function. Optimization of a well-defined function is solvable through standard deterministic techniques. However, we often need to make decisions under uncertainty. In this case, we often cannot predict the exact outcome of a decision because the outcome depends on unknown factors. This leads to the concept of fuzzy optimization, which describes mathematical programming problems in which the functional relationship between the decision variables and the objective function is fuzzy (or known linguistically) [Carlsson et al., 1998].

First, we will give an overview of fuzzy optimization. Then, we will present some applications of fuzzy optimization and investigate ways to extend the methodology to risk assessment.

Keywords: Fuzzy logic, Optimization, Risk assessment

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## I. Introduction

The residential housing market catastrophe largely contributes, among other events, to the global financial crisis of 2008 (Segal, 2011:330). This crisis has prompted policy makers and practitioners to question the efficacy of the existing standard for risk assessment and policies[1].

Risk is commonly associated with unexpected event. The terrorist attacks of September 11, 2001, the August 2005 Hurricane Katrina in New Orleans[2], and the April 2010 massive BP oil spill in the Gulf of Mexico are examples of such unexpected events. Knight (1921) started the discussion about risk and uncertainty in his now classic book "Risk, uncertainty, and profit". Generally, risk refers to the possibility (or the possible consequences) of things going wrong (Panjer, 2006: 3). Better et al. (2008)[3] and Ostrom and Wilhelmsen (2012: 6) described risk as the probability of an unexpected event that results in negative consequences. In many cases huge amounts of money are involved. Risk also involves exposure. Holton (1997; 2004) defines risk as exposure to uncertainty, and hence views uncertainty and exposure to that uncertainty as the risk's components. Often, a focus for policy makers and reinsurance companies is on low probability events with high consequences that lead to considerable damage, loss, death, and environmental impairment for example (Brillinger, 2002). From a risk assessment perspective, risk may also involve unexpected events with positive consequences for the firm. Ignoring this side of risk leads to incomplete assessment, because downside and upside events may offset each other. Therefore, a complete definition is of the type given by Segal (2011:19): risk is uncertainty, deviation from expected, and includes both positive and negative deviation.

Generally, risk events are grouped within the following risk categories: hazard risk, financial risk, strategic risk, and operational risk. A detailed definition and the composition of these groups are provided by Segal (2011:116). Independently of the risk categories, a risk assessment (RA) is a systematic process for identifying and evaluating potential risks and opportunities that could positively or negatively affect the achievement of an enterprise's objectives (Price-Waterhouse-Cooper, 2008). Generally, the RA process follows three steps: identification, analysis, and evaluation of risk. Detailed descriptions of these steps are provided in the literature (Segal (2011, p. 113) and Price-Waterhouse-Cooper (2008)). The flowchart in Figure 1 gives the components of the RA process.

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[1] Louise Francis challenged the regulators' position towards a "fraud friendly environment" in Francis, L., "The Financial Crises: Why Won't We use the F(raud) Word?", SOA 2011, 44:47.

[2] US Army Engineers estimated that Hurricane Katrina was a 1-in-396-year event (Segal, 2011: 9)

[3] The term opportunity is used for an unexpected event that would have a positive impact (Better et al. 2008)

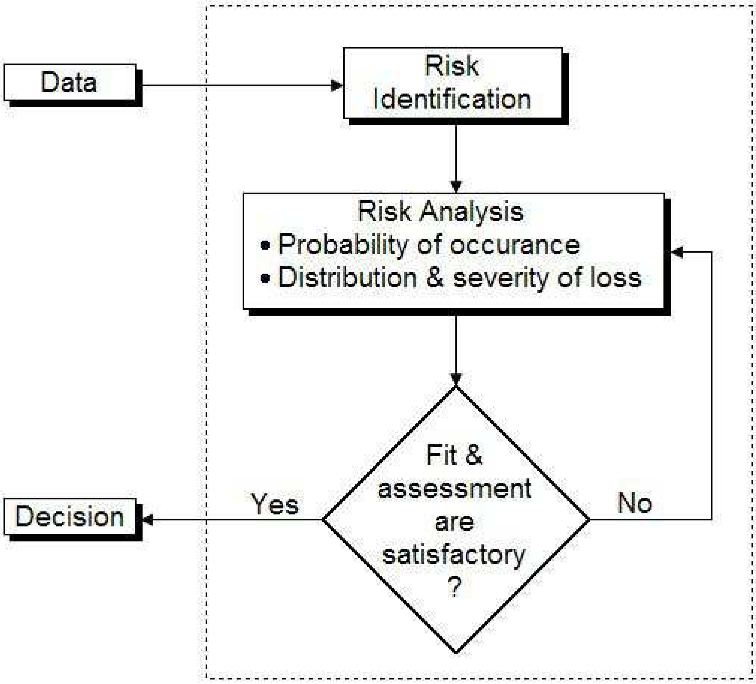


Figure 1: Risk Assessment Flowchart

A risk assessment process aims at providing a clear picture of what we know regarding the nature of a particular risk and the degree of uncertainty surrounding any estimates. Kunreuther (2002) suggested using an exceedance probability (EP) curve to measure experts' knowledge (or lack of knowledge) and projection about a risk event. An EP curve shows the probabilities that certain level of losses will be exceeded. Figure 2 is an illustration of the EP curve for dollar losses to homes in Los Angeles from an earthquake [i] (Kunreuther, 2002).

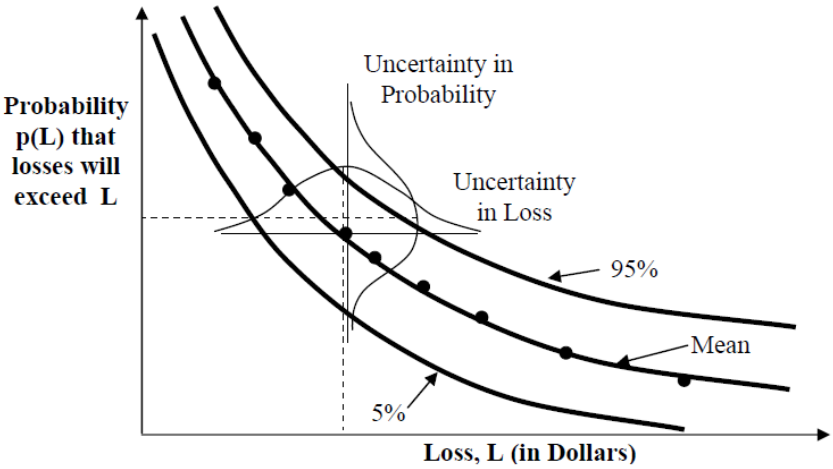


Figure 2: Exceedance probability (EP) curve
(From Kunreuther, 2002)

## II. Standard Optimization as a Risk Assessment Tool

Various qualitative and quantitative risk assessment techniques have been developed to help risk managers achieve their objective. Optimization is an example of such a quantitative technique, and includes many branches, such as linear / non-linear optimization, convex optimization, conic optimization, global optimization, discrete optimization, etc[4]. Optimization consists in finding an optimal solution to a given problem, under given circumstances. Progress in computational technique has made optimization more accessible and easier to use, hence more attractive to a larger number of practitioners. Optimization, as a powerful modelling and problem solving methodology, has a broad range of applications: in nature for example, physical systems tend to a state of minimum energy, while molecules in an isolated chemical system react with each other until the total potential energy of their electrons is minimized, and rays of light follow paths that minimize their travel time (Nocedal and Wright, 2000). Areas of application of optimization also include engineering[5] (transportation, production planning, design and data fitting), industry [6] (where for example airline companies schedule crews and aircraft to minimize cost aerospace), and management science. Optimization is also widely used in insurance and actuarial sciences where, for example, investors will seek to build up portfolios that minimize risks while achieving a high rate of return. Brockett and Xia (1995) give a review of operational research in insurance.

In the next section we the review the mathematical formulation of an optimization problem, provide an example of optimization in insurance and actuarial sciences, and then present an application of optimization to risk assessment.

### 2.1 Introduction to Standard (Crispy) Optimization

The process of optimization aims at finding the optimal solution to a given problem. Generally, an optimization problem is of the form (Boyd and Vandenberghe, 2009):

$$\begin{aligned} &\text{Minimize} \quad f_0(x) \\ &\text{Subject to} \quad f_i(x) \leq b_i \quad \text{for } i = 1, \dots, m \end{aligned} \tag{1}$$

[4] Nocedal, J. and Wright, S., 2000, "Numerical Optimization", Springer Series in Operations Research and Financial Engineering, 2nd ed. provide a survey of the various branches in optimization.

[5] Rao, S., 2009, "Engineering Optimization: Theory and Practice", John Wiley & Sons, 4th ed., present engineering applications of optimization.

[6] Ciriany, T., and Robert C. Leachman, R., 1993, "Optimization in industry" provide a survey of applications of optimization in industry.

where $x = (x_1, \ldots, x_n)$ is the optimization variable, $f_0 : R^n \rightarrow R$ is the objective function, and the functions $f_i : R^n \rightarrow R$, $i = 1, \ldots, m$ are the set of inequality constraints. An optimal solution $x^*$ of the optimization (1) satisfies: for any variable z with $f_1(z) \leq b_1, \ldots, f_m(z) \leq b_m$, we have $f_0(z) \geq f_0(x^\star)$. Depending on the properties of the objective function and the constraints, optimization (1) can be a linear, quadratic, discrete, or a convex optimization problem.

Problem (1) is a linear optimization if the objective and the constraint functions are linear:

$$f_i(\alpha x + \beta y) = \alpha f_i(x) + \beta f_i(y) \qquad \text{for } i = 0, 1, \ldots, m \qquad (2)$$

for all $x, y \in R^n$ and all $\alpha$, $\beta \in R$.

We obtain a convex optimization when the objective and the constraint functions are convex:

$$f_i(\alpha x + \beta y) \leq \alpha f_i(x) + \beta f_i(y) \qquad \text{for } i = 0, 1, \ldots, m \qquad (3)$$

for all $x, y \in R^n$ and all $\alpha$, $\beta \in R$. The parameters satisfy: $\alpha + \beta = 1$, $\alpha \geq 0$, $\beta \geq 0$

A quadratic problem is obtained when the objective function is quadratic. The constraints are usually linear functions (Rachev et al., 2008).

$$f_0(x) = c^T x + ½ x^T H x \qquad \text{for all } x \in R^n \qquad (4)$$

where $c = (c_1, \ldots, c_n)$ is a vector of coefficients for the linear part of the objective function and $H=\{c_{ij}, \ i, j = 1, \ldots, n\}$ is a $n$ x $n$ matrix defining the quadratic part of the objective. Shapiro (1986) provided an extensive survey of operational research methodologies, and their applications in insurance. For example, areas of application of quadratic optimization include portfolio analysis and international re-insurance.

When there is no constraint on the set of feasible solution, Equation (1) becomes an unconstrained optimization. An unconstrained problem can be solved using the first-order and the second-order conditions[7] on the function gradient ($\nabla f(x)$) and the Hessian matrix (H) of second order derivatives (see Appendix A-1). For constrained optimization, the method of Lagrange multipliers is often applied (Details of which can be found in Boyd and Vandenberghe, 2009; Rachev, 2008).

[7] The first-order condition $\nabla f(x) = 0$ is a necessary condition for finding a function extrema (minimum/maximum), and the second-order condition (sign of the Hessian) provides a sufficient additional condition for the extremum. (Details can be found in Boyd and Vandenberghe, 2009; Rachev, 2008)

### 2.2 Applications of Optimization in Insurance

Operation research in general and optimization in particular have a long history in insurance and actuarial sciences. In the applications to insurance, the objective function $f_0(x)$ in Eq. (1) can be such things as the expected value of the probability distribution of the throughput (the average quantity of non-defective parts produced per unit time) at a factory, or the fifth percentile of the distribution of the net present value (NPV) of a portfolio of investments (Better et al., 2008).

Brockett and Xia (1995) give a detailed and technical review of applications of optimization in insurance. For example, linear optimization (adapted from Schleef, 1989) is used to measure the cost of whole life insurance. The objective function represents the discount cash flows associated with the selected policy, and needs to be maximized. The constraints are the amount that the insured is willing to pay for his policy, and the required yearly level of protection. The linear programming corresponding to this problem is as follows.

$$
\begin{aligned}
&\text{Maximize} \quad \frac{C_n}{(1+i)^n}u + \sum_{i=1}^{n}\frac{(w_t - z_t)}{(1+i)^{(t-1)}} \\
&\text{subject to} \quad P_t u + w_t - z_t \le b_t, \quad \text{for } t = 1,\cdots,n; \\
&\qquad u + \sum_{i=1}^{n}\frac{(w_t - z_t)}{(1+i)^{(t-1)}} \ge I_j \quad \text{for } j = 1,\cdots,n; \text{ and} \\
&\qquad u, w_t, z_t \quad \text{are non-negative,}
\end{aligned}
\tag{5}
$$

where $w_t$, $z_t$, and u are decision variables; $w_t$ is the amount lent externally by the insured at the beginning of year t; $z_t$ is the amount borrowed externally by the insured at the beginning of year t; u is the face value of insurance purchased at the time t=0; $P_t$ is the net premium rate in year t; $C_t$ is the cash-value rate at the end of year t; $b_t$ is the amount budgeted by the insured at the beginning of year t; $I_t$ is the insurance protection required at the beginning of year t; and $n$ is the number of years in the planning period.

Recently, Dhaene et al. (2012) use optimization to solve a capital allocation problem. The problem was formulated in the form of minimum distance, and a solution was obtained by minimizing the weighted sum of measure for the deviations of the business unit's losses from their respective capitals.

## 2.3. Optimization as a Risk Assessment Tool

In practice risk analysis often consists of (i) finding the probability of occurrence of an event, and (ii) computing the statistical distribution of the (potential) damages. As a result of this process, graphics or hazard risk maps can be obtained and used for prediction (Brillinger, 2002).

We present an example based on Better et al. (2008). An insurer has a number of potential projects for which the revenues for a horizon of approximately *n* periods (depending on the project) are given as probability distributions. For each project, there is an initial investment and a number of business development, engineering and earth sciences personnel. As a constraint, there is a budget limit for the investments, and a limited number of personnel of each skill category. A probability of success by project is also assigned. Following the authors, we assume without loss of generality that Project A has a probability of success 0.6. There is a window of opportunity for each project, which may start in different time periods. The insurer aims at selecting a set of optimal projects to invest in that will best further its corporate goals.

The authors compare three risk assessment methodologies namely the mean-variance approach (Markowitz, 1952), the 5th percentile, and the value-at-risk (VaR) that were all implemented through optimization. The main optimization problem is of quadratic form:

$$\begin{aligned} &\text{Maximize} \quad r^{\mathrm{T}} w - k w^{T} Q w, \\ &\text{subject to} \quad \sum_{i=1}^{n} c_i w_i = b \quad \text{and} \\ &\qquad\qquad w_i \in \{0,1\} \end{aligned} \tag{6}$$

where r is a vector of portfolio returns, $Q$ is a covariance matrix of returns, the coefficient $k$ describes the insurer's risk aversion, the constant $c_i$ represents the initial investment in project $i$, the term $w_i$ is a binary variable representing the decision whether to invest in project $i$, and the constant $b$ is the available budget.

With the mean-variance approach, the optimization problem is as follows (assuming that the objective is to maximize the expected NPV of the portfolio, while keeping the standard deviation of the NPV below a specified threshold of for example \$140M):

$$
\begin{aligned}
\text{Maximize} \quad & \mu_{\text{NPV}} \quad \text{(objective function)},\\
\text{Subject to} \quad & \sigma_{\text{NPV}} < \$140\text{M} \quad \text{(requirement)},\\
& \sum_i c_i x_i \le b \quad \text{(budget constraint)},\\
& \sum_i p_{ij} x_i \le P_j \;\; \forall j \quad \text{(personnel constraints)},\\
& \text{All projects must start in year 1},\\
& x_i \in \{0,1\} \;\; \forall i \quad \text{(binary decisions)}.
\end{aligned} \tag{7}
$$

When the risk is controlled by the 5th percentile (assuming that the objective is to maximize the average return, as long as no more than 5% of the trial observations fall below a level of $176M for example), the optimization problem is as follows:

$$
\begin{aligned}
\text{Maximize} \quad & \mu_{\text{NPV}},\\
\text{Subject to} \quad & P(5)_{\text{NPV}} > \$176\text{M} \quad \text{(requirement)},\\
& \sum_i c_i x_i \le b \quad \text{(budget constraint)},\\
& \sum_i p_{ij} x_i \le P_j \;\; \forall j \quad \text{(personnel constraints)},\\
& \sum_{m_i \in M} x_i \le 1 \;\; \forall i \quad \text{(mutual exclusivity)},\\
& x_i \in \{0,1\} \;\; \forall I \quad \text{(binary decisions)},
\end{aligned} \tag{8}
$$

We often need to make decisions under uncertainty. In this case, we often cannot predict the exact outcome of a decision because the outcome depends on unknown factors. This leads to the concept of fuzzy optimization, which describes mathematical programming problems in which the functional relationship between the decision variables and the objective function is fuzzy (or known linguistically) [Carlsson et al., 1998]

## III. Fuzzy Optimization as a Risk Assessment Tool

Zadeh (1965) introduced the notion of fuzzy logic, which has since then gained recognition and was intensively applied in mathematics and computer sciences (Dubois and Prade, 1980; Kandel, 1986; Zimmerman, 1996). Various applications of fuzzy logic exist in the insurance literature, especially in insurance underwriting (DeWit, 1982), classification of insurance risk (Ebanks et al., 1992; Derrig and Ostaszewski, 1995), projected liabilities (Cummins and Derrig, 1993;

Sanchez and Gomez, 2003), future and present values (Buckley, 1987) and finance (Lemaire, 1990). Ostaszewski (1993) reviewed some applications to risk theory and Shapiro (2004) provided an extensive overview of the possible applications of fuzzy logic in insurance.

### 3.1 Introduction to Fuzzy Logic

According to Zadeh, fuzzy logic has four facets (Shapiro, 2004): (1) the logical facet which deals with approximate reasoning, (2) the set theory facet which involves classes having unsharp boundaries, (3) the relational facet which deals with linguistic variables, and (4) the epistemic facet which is concerned with knowledge, meaning, and linguistic. The studies we reviewed involve the first three facets.

Fuzzy numbers are characterized by their membership functions which can be triangular, trapezoidal, Gaussian, generalized bell or a combination of these basic classes (Shapiro, 2004). A membership function describes the grade of membership of each variable. As an example, Figure 3 shows the membership function for a set of clients with high risk capacity: individuals with a risk capacity of 50 percent, or less, are assigned a membership grade of zero and those with a risk capacity of 80 percent, or more, are assigned a grade of one. Between those risk capacities, (50%, 80%), the risk capacity category is fuzzy.

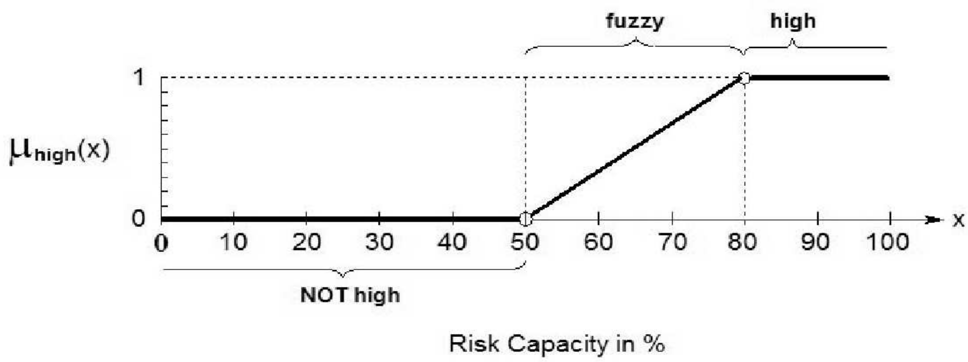


Figure 3 (From Shapiro, 2004):
Membership function for set of clients with high risk capacity

### 3.2 Fuzzy Optimization

Bellman and Zadeh (1970) introduce the concept of fuzzy optimization in their seminal paper "Decision Making in a Fuzzy Environment". Fuzzy optimization differs from classical optimization in the sense that the objective function and constraints are given the same importance: objective function and constraints are both (written in term of membership functions if appropriate and) optimized simultaneously. The membership functions are linked by a linguistic conjunction: "and" (for maximization) and "or" (for minimization).

For a maximization problem, where the optimal decision is the option with the highest degree of membership in the decision set, the optimal solution is by

$$x^* = \max [\min\{\mu_G(x), \mu_C(x)\}] \quad (9)$$

If the optimal decision is the option with the lowest degree of membership in the decision set, then the optimal solution is as follows

$$x^* = \min [\max\{\mu_G(x), \mu_C(x)\}] \quad (10)$$

A representation of the relationship between the MFs for the goal G, the constraint C and the decision D is given in Fig. 4 (Shapiro, 2004, p.402).

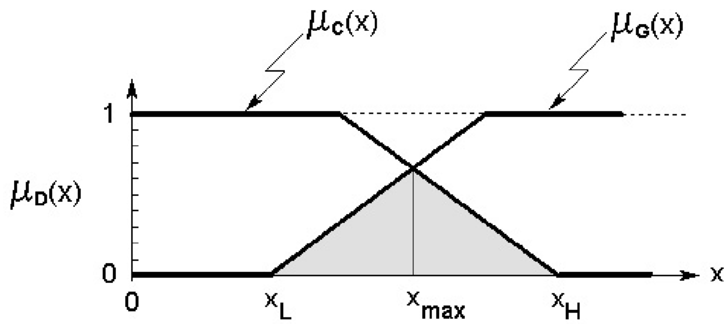


Figure 4: Decision Making in Fuzzy Optimization

A fuzzy linear programming can be reformulated as an equivalent crispy optimization problem and the standard optimization tools can be used to obtain the optimal solution (see Brockett and Xia, 1995; Shapiro, 2004). For example, given the following fuzzy linear programming (Zimmermann 1996: 289)

$$\begin{aligned} &C = \sum_{ij} c_{ij}\, x_{ij} \,\tilde{\leq}\, C_0 && \text{Objective} \\ &z_i = \sum_{j} a_{ij}\, x_{ij} \,\tilde{\geq}\, b_i \quad , \quad x_{ij} \geq 0 && \text{constraints} \end{aligned} \quad (1)$$

where $C_0$ is the aspiration level for the objective function, and the coefficients $a_{ij}$, $b_i$, and $c_{ij}$ are not necessarily crisp numbers. This fuzzy linear programming (LP) problem can be resolved by reformulating it as a crisp LP problem. The essence of one approach[8] to doing this is depicted in Figure .

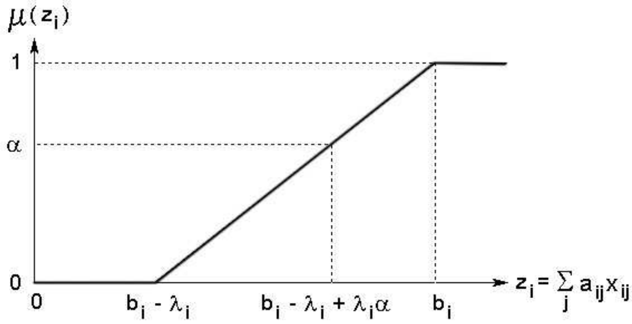


Figure 5: Equivalent Crisp Constraint

As indicated, $z_i$ is a fuzzy number, whose membership function is zero for $z_i \le b_i - \lambda_i$, one for $z_i \ge b_i$, and linearly increasing in the interval. Zimmermann refers to $\lambda$ as a tolerance interval. Using an $\alpha$–cut to provide a minimum acceptable satisfaction level, that is, $\mu(z_i) \ge \alpha$ is an acceptable constraint, we see from the diagram that an equivalent constraint is $z_i \ge b_i - \lambda_i + \lambda_i \alpha$. Similarly, $C \le C_0 + \lambda - \lambda \alpha$.

Thus, given the values of $\lambda$, the equivalent crisp programming problem becomes one of maximizing $\alpha$ subject to the equivalent constraints, that is:

$$\text{Maximize:} \quad \alpha \qquad (2)$$

$$\text{Subject to:} \quad z_i - \lambda_i \alpha \ge b_i - \lambda_i ;$$

$$C + \lambda \alpha \le C_0 + \lambda \text{ ; and}$$

$$0 \le \alpha \le 1.$$

[8] Adapted from Brockett and Xia (1995), pp. 34-38.

**Basic steps in fuzzy optimization problem**

Generally, a fuzzy optimization problem can be solved through the following steps (Bellman and Zadeh, 1970; DaSilva et al., 2002):

1. Fuzzification of the objective function: the membership associated with the objective function f(x) is computed through the following formula:

$$\mu_f(x) = \frac{f(x) - 1}{\max(f(x)) - \min(f(x))} \qquad (11)$$

   Note: max(f(x)) and min(f(x)) represent the maximum and minimum values in the feasible interval for the function f(x).

2. Fuzzification of the constraints is done the same way.

3. Membership of the optimal function: the membership functions for all the constraints and the objective function are combined, and the decision making formulas (9) and (10) are used depending on whether we have a minimization or a maximization problem.

**Example:** The following example is adapted from DaSilva et al. (2002): Consider a (non-linear) quadratic function f(x) and let's assume that we aim at maximizing f under the given fuzzy constraints below.

$$\text{Max } f(x) = 10 - x - 25/x^2$$

Constraints:
$C_1$: x must be equal or greater than 2 and equal or less than 6 = { $2 \leq x \leq 6$ }
$C_2$: a good value for x is equal or less than 3 and an acceptable value is not much greater.

Solution: we will follow the three steps described above.

1) Fuzzification of the objective function

$f(2)$ =1.75 (min in [2,6]); $f(3.68)$ = 4.47= (max of f in [2,6]);

$$F(x) = \mu_f(x) = \frac{f(x) - 1}{\max(f(x)) - \min(f(x))} = 3.03 - (x/2.72) - (9.19/x^2)$$

2) The constraints are equivalent to the following membership

$$C_1(x) = \begin{cases} 0 & \text{for } x < 2 \\ 1 & \text{for } 2 \leq x \leq 6 \\ 0 & \text{for } x > 6 \end{cases} \qquad C_2(x) = \begin{cases} 1 & \text{for } x \leq 3 \\ \frac{1}{3} + \frac{2}{x} & \text{for } x > 3 \end{cases}$$

By merging both constraints $C_1$ and $C_2$, we obtain the followingf function

$$C(x) = \begin{cases} 0, & x < 2 \\ 1, & 2 < x < 3 \\ \frac{1}{3} + \frac{2}{x}, & 3 < x < 6 \end{cases}$$

3) The graphs of the objective function and the combined constraints is then obtained

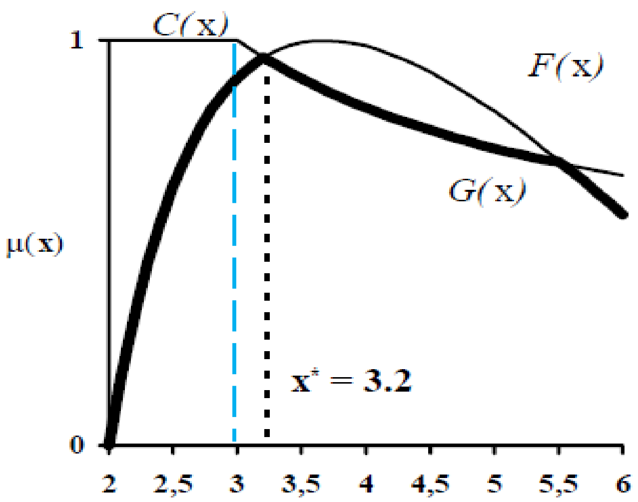


Figure 6: Optimal Solution
(Adapted from DaSilva et al., 2002)

### 3.3 Fuzzy Optimization as RA tool

This section is based on the paper by Liu and Zhang (2012) which deals with the risk assessment of natural disasters and sporadically occurring events in general, and on risk analysis and prediction for tropical cyclones in particular. The purpose of this risk analyses is to identify a functional relationship between the probability distributions of hazard causes (e.g., rainfall amounts or surface wind strengths) and hazard impacts (e.g., on human-beings, buildings, crops) using information matrices (i.e., inputs vs. outputs). The hazard risk assessment model is done through the steps below, which are summarized below.

Steps in hazard risk assessment (Liu and Zhang, 2012)

1) Start with available data: the hazard-impact indicator matrix $x_{ij}$ (i = 1, 2, . . . , 5; j = 1, 2, . . . , 17), from the 17-year sample data.

2) Standardize the hazard-impact indicator matrix by applying a scaling factor

$$r_{ij} = \frac{x_{ij}}{\max\limits_{1\le j\le 17}\{x_{ij}\} + \min\limits_{1\le j\le 17}\{x_{ij}\}}, \quad i = 1, 2, \ldots, 5,$$

3) We obtain the fuzzy weighted hazard-impact indicator matrix

$$\mathbf{R} = (r_{ij})_{5\times 17}$$

4) We determine the categorical weights of the hazard-impact indicators, using a genetic projection pursuit algorithm.
   a. For this we compute

$$z_j = \sum_{i=1}^{5} e_i r_{ij} \quad (j = 1, 2, \ldots, 17)$$

   b. Then, $S_z$ =standard deviation (z)
   c. And the local density ($D_z$) is given by

$$D_Z = \sum_{i=1}^{5}\sum_{j=1}^{17}(K - d_{ij})u(t)(K - d_{ij})$$

   where K = 0.1$S_z$, distance $d_{ij} = |z_i - z_j|$, t = K - $d_{ij}$, and the unit step function u(t) is 0 for t < 0 and 1 for t≥0

5) The associated objective function after projection is

$$Q_e = S_z D_z.$$

6) The projection directions reflect the characteristics of hazard-impact and cause indicators. So the optimized projection direction of $Q_e$ is obtained by maximizing the projected objective function

$$\max Q_e = S_Z D_Z,$$
$$\text{s.t.} \quad \sum_{i-1}^{5} e_i^2 = 1, \quad \text{and} \quad e_i \ge 0$$

7) Determine the ranking weights of the hazard-impact indicators, using the GAHP method

$$s_i = \sum_{j=1}^{17} r_{ij},$$

   and from the following expression, the C are obtained.

$$\begin{cases} \frac{s_i - s_j}{max\{s_i\} - \min\{s_i\}}(c_m - 1) + 1, & s_i \geq s_j, \\ \left(\frac{s_j - s_i}{\max\{s_i\} - \min\{s_i\}}(c_m - 1) + 1\right)^{-1}, & s_i < s_j, \end{cases}$$

8) Determine the combined weights between the categorical and ranking weights, and solve the fuzzy optimization problem that follows,

$$\min\ F = \sum_{j=1}^{5} \sum_{i=1}^{5} (\mu|a_{i1} - a_i|r_{ij} + (1 - \mu)|a_{i2} - a_i|r_{ij}),$$

$$\text{s.t.} \quad \sum_{i=1}^{5} a_i = 1, \quad \text{and } a_i \geq 0; \quad i = 1, 2, \ldots, 5,$$

to obtain the combined weights of the hazard-impact indicators (HIIS).

**Conclusion**

The aim of this paper was to make an introduction to the application of fuzzy optimization in risk assessment. We present some applications of standard optimization in risk assessment. Fuzzy optimization differs from classical optimization in the sense that the objective function and constraints are given the same importance. In the final example presented, fuzzy optimization was applied to hazard risk assessment in order to identify a functional relationship between the probability distributions of hazard causes and the hazard impacts.

## Appendices

### A-1

The gradient ($\nabla f(x)$) and the Hessian matrix (H) of second order derivatives of a function $f(x) = f(x_1, x_2, \cdots, x_n)$ are as follows.

$$\nabla f(x) = \left( \frac{\partial f(x)}{\partial x_1}, \cdots, \frac{\partial f(x)}{\partial x_2} \right)$$

$$\mathrm{H}(x) = \nabla^2 f(x) = \begin{pmatrix} \frac{\partial^2 f(x)}{\partial_2 x_1} & \frac{\partial^2 f(x)}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f(x)}{\partial x_2 \partial x_1} & \frac{\partial^2 f(x)}{\partial^2 x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f(x)}{\partial x_n \partial x_1} & \frac{\partial^2 f(x)}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial^2 x_n} \end{pmatrix}$$

## References


Better, M., Glover, F., Kochenberger, G., Wang, H., 2008, Simulation Optimization Applications in Risk Management. International Journal of Information Technology and Decision Making, 7(4), 571-587

Boyd and Vandenberghe, 2009, Convex Optimization, Cambridge University Press, $7^{th}$ ed.

Brillinger, D.,R., 2002, Some Examples of Risk Analysis in Environmental Problems, University of California, Berkeley.

Carlsson, C. et al. (1998) Optimization under fuzzy rule constraints. The Belgian Journal of Operational Research, Statistics and Computer Science, 38, 17-24.

DaSilva, L.E, Torres, G., L., Dos Reis, L., O., 2002, “Application of Fuzzy Optimization in Energy Saving”. Rev. Ciênc. Exatas, Taubaté, v. 5-8, p. 21- 35.

Francis, L., 2011, “The Financial Crises: Why Won’t We use the F(raud) Word?”, SOA 2011, 44:47.

Holton, G., 1997, “Subjective Value at Risk”, Financial Engineering News, 1(Aug). Reprinted by SOA, 1998, “Risks and Rewards Newsletter” October 1998 – Issue No. 31.

Holton, G., 2004, “Defining Risk”, Financial Analysts Journal 60 (6): 19-25.

Knight, F. H., 1921, Risk, Uncertainty, and Profit. Boston: Houghton Mifflin, 1921.

Kunreuther, H., 2002, “Risk Analysis and Risk Management in an Uncertain World”, Risk Analysis, 22 (4): 655-664

Liu, H., and Zhang, D.-L., 2012, “Analysis and prediction of hazard risks caused by tropical cyclones in Southern China with fuzzy mathematical and grey models”, Applied mathematics Modelling, 36: 626-637.

Markowitz, H.M., “Portfolio Selection”, Journal of Finance 7 (1952): 77-91.

Ostrom, L., and Wilhelmsen, C., 2012, Risk assessment: tool, techniques, and their applications, Wiley & Sons.

Panjer, H., 2006, “Operational Risk: Modeling Analytics”, Wiley Series in Probability and Statistics.

Price-Waterhouse-Cooper, 2008. “A Practical Guide to Risk Assessment” www.pwc.com/en_US/us/issues/enterprise-risk-management/assets/risk_assessment_ guide.pdf

Rachev, S., Stoyanov, S., Fabozzi, F., 2008, Advanced Stochastic Models, Risk Assessment, and Portfolio Optimization: The Ideal Risk, Uncertainty, and Performance Measures, Wiley.

Segal, 2011, Corporate Value of Enterprise Risk Management - The next Step in Business Management, Wiley Corporate F&A.

Shapiro, A. F., 1986. “Applications of Operations Research Techniques in Insurance”, in Insurance and Risk Theory, ed. M. Goovaerts, E de Vylder, and J. Haezendonck. Norwell, Mass.: Kluwer Academic Publishers.

Shapiro, A. F., 2004, “Fuzzy Logic in Insurance”, Insurance Mathematics and Economics, 35: 399-424

Shapiro, A. F., 2007, “An Overview of Insurance Uses of Fuzzy Logic”, in Computational Intelligence in Economics and Finance, Vol. II

Sweeting, P., 2011, Financial Enterprise Risk Management, International Series on Actuarial Science, Cambridge University Press.

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[i] This EP curve is obtained by first combining the set of events that could produce a given dollar loss. Then, the resulting probabilities of exceeding losses of different magnitudes are found. Based on these estimates, the mean EP is computed. The uncertainty associated with the probability of an event occurring and the magnitude of dollar losses is reflected in the 5% and 95% confidence interval curves (Kunreuther, 2002).