Discounted compound renewal sums with a stochastic force of interest

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Abstract

Recursive moments, moments generating functions, distributions functions and risk measures have been found for the compound renewal sums with discounted claims, for a constant force of real interest.

In this talk we present several results on the (joint) moments, on the (joint) moments generating functions and on regression aspects of these discounted renewal sums, in a context that may involve a stochastic force of real interest. Examples will be given for the counting Poisson process and for the Ho-Lee-Merton interest rate model.

Keywords : Compound Poisson process; discounted aggregate claims; force of interest; Itô process; joint moments; renewal process; stochastic interest rate.

Our risk model

(i) The claims counting processes $\{N(t), t \geq 0\}$ and $\{N_d(t), t \geq 0\}$ form respectively an ordinary and a delayed renewal process, and for $k \in \mathbb{N} = \{1, 2, 3, ...\}$:

- the positive claim occurrence times are given by $\left\{T_k, k \in \mathbb{N}\right\}$,
- the positive claim inter-arrival times are given by $\boldsymbol{\tau}_k = T_k - T_{k-1}$, with $T_0 = 0$.

(ii) The corresponding deflated claim severities $\left\{X_k, k \in \mathbb{N}\right\}$ are such that

- $\left\{X_k, k \in \mathbb{N}\right\}$ are i.i.d. .
- $\left\{X_k, \boldsymbol{\tau}_k ; k \in \mathbb{N}\right\}$ are mutually independent.
- The m.g.f. of X_1 exists in a neighourhood $\Omega \subset \mathbb{R}$ containing zero.

(iii) The aggregate discounted value at time 0 of the inflated claims recorded over the period $\left[0,t\right]$ are given respectively, for the ordinary and the delayed renewal case, by

$$Z\left(t\right)=\sum_{k=1}^{N(t)} D\left(T_k\right)X_k \quad , \quad Z_d\left(t\right)=\sum_{k=1}^{N_d(t)} D\left(T_k\right)X_k \quad ,$$

where

$$Z\left(t\right)=Z_d\left(t\right)=0 \text{ if } N\left(t\right)=N_d\left(t\right)=0, \quad D\left(T_k\right)=\exp\left\{-\int_0^{T_k}\delta\left(x\right)dx\right\} ,$$

and $\delta\left(x\right)$ is the force of real interest, which can be a deterministic function or a random variable.

Remark : We will note $Z_o\left(t\right)$ for the risk process generated by an embedded ordinary renewal process.

A reminder

Moments of compound renewal sums with discounted claims have been considered for the first time by Léveillé and Garrido (2001), for a positive constant force of real interest. Using essentially renewal arguments, these recursives formulas have been obtained :

- for the ordinary renewal case

$$E\left[Z^n(t)\right]=\sum_{k=0}^{n-1}\binom{n}{k}E\left[X^{n-k}\right]\int_0^t e^{-n\delta v}E\left[Z^k(t-v)\right]dm(v)\ \ ,\ \ m(t)=E\left[N(t)\right].$$

- for the delayed renewal case

$$E\left[Z_d^n(t)\right]=\sum_{k=0}^{n-1}\binom{n}{k}E\left[X^{n-k}\right]\int_0^t e^{-n\delta v}E\left[Z_o^k(t-v)\right]dm_d(v)\ \ ,\ \ m_d(t)=E\left[N_d(t)\right].$$

Recursive joint moments for a constant δ

We need first a lemma in order to get these recursive joint moments.

Lemma 1 : Consider an ordinary or a delayed renewal counting process, such as defined previously. Then, for any $t>0$, $h>0$, $\delta \geq 0$ and $(u,v)\in\Omega\times\Omega$, the joint m.g.f. of our risk process satisfies respectively the following integral equations:

(1) For the ordinary renewal case :

$$M_{Z(t),Z(t+h)}(u,v)=\overline{F}_{\tau_1}(t+h)+\int_t^{t+h} M_X\left(ve^{-\delta x}\right)M_{Z(t+h-x)}\left(ve^{-\delta x}\right)dF_{\tau_1}(x)$$

$$+\int_0^t M_X\left((u+v)e^{-\delta x}\right)M_{Z(t-x),Z(t+h-x)}\left(ue^{-\delta x},ve^{-\delta x}\right)dF_{\tau_1}(x)\ .$$

(2) For the delayed renewal case :

$$M_{Z_d(t),\, Z_d(t+h)}(u,v) = \bar{F}_{\tau_1}(t+h) + \int_t^{t+h} M_X\left(ve^{-\delta x}\right) M_{Z_o(t+h-x)}\left(ve^{-\delta x}\right) dF_{\tau_1}(x)$$

$$+\int_0^t M_X\left((u+v)e^{-\delta x}\right) M_{Z_o(t-x),\, Z_o(t+h-x)}\left(ue^{-\delta x}, ve^{-\delta x}\right) dF_{\tau_1}(x) \; .$$

where $\bar{F}_{\tau_1}(t) = 1 - F_{\tau_1}(t)$.

Proof of (1): We condition first on $N(t)$, $N(t+h)$, $T_1,\ldots,T_{N(t+h)}$, which yields

$$M_{Z(t),\, Z(t+h)}(u,v) = E\left[\prod_{k=1}^{N(t)} M_X\left((u+v)e^{-\delta T_k}\right) \prod_{k=N(t)+1}^{N(t+h)} M_X\left(ve^{-\delta T_k}\right)\right],$$

and thereafter we condition on τ_1 to get the result.

Theorem 1 : According to the hypotheses of lemma 1, the joint moments of our risk process are given respectively, for $n, m \in \mathbb{N}$, by :

(1) For the ordinary renewal case :

$$E\left[Z^n(t)\,Z^m(t+h)\right]=\sum_{k=1}^{n+m}E\left[X_1^k\right]\sum_{i=[k-m]_+}^{\min(k,n)}\binom{n}{i}\binom{m}{k-i}$$
$$\times\int_0^t e^{-(n+m)\delta u}E\left[Z^{n-i}(t-u)\,Z^{m-(k-i)}(t+h-u)\right]dm(u)\,.$$

(2) For the delayed renewal case :

$$E\left[Z_d^n(t)\,Z_d^m(t+h)\right]=\sum_{k=1}^{n+m}E\left[X_1^k\right]\sum_{i=[k-m]_+}^{\min(k,n)}\binom{n}{i}\binom{m}{k-i}$$
$$\times\int_0^t e^{-(n+m)\delta u}E\left[Z_o^{n-i}(t-u)\,Z_o^{m-(k-i)}(t+h-u)\right]dm_d(u).$$

Proof : The preceding equations follow directly by taking the appropriate partial derivatives of the integral equations of lemma 1, a number of times with respect to u and with respect to v, then each time evaluating these expressions at $(u,v)=(0,0)$ and thereafter using induction. □

Remark 1 : (1) If we set $h=0$ in the equations of theorem 1, then we retrieve the preceding recursive expressions for the moments.

(2) For $n=m=1$, the joint moments of the risk process of theorem 1 can be written for the ordinary renewal case as follows

$$E\left[Z(t)Z(t+h)\right]=E\left[X_1\right]\int_0^t e^{-2\delta u}\left\{E\left[Z(t-u)\right]+E\left[Z(t+h-u)\right]\right\}dm(u)$$
$$+E\left[X_1^2\right]\int_0^t e^{-2\delta u}dm(u)$$

which is equivalent to

$$E\left[Z(t)Z(t+h)\right]=E\left[Z^2(t)\right]+E^2\left[X_1\right]\int_0^t\int_{t-u}^{t+h-u} e^{-\delta(v+2u)}\ dm(v)dm(u)\ ,$$

and similarly for the delayed renewal case

$$E\left[Z_d(t)Z_d(t+h)\right]=E\left[X_1\right]\int_0^t e^{-2\delta u}\left\{E\left[Z_o(t-u)\right]+E\left[Z_o(t+h-u)\right]\right\}dm_d(u)$$
$$+E\left[X_1^2\right]\int_0^t e^{-2\delta u}dm_d(u)$$

which yields

$$E\left[Z_d(t)Z_d(t+h)\right]=E\left[Z_d^2(t)\right]+E^2\left[X_1\right]\int_0^t\int_{t-u}^{t+h-u} e^{-\delta(v+2u)}\ dm_o(v)dm_d(u)\ .$$

Example 1 : Consider a constant force of real interest $\delta > 0$ and a counting Poisson process with parameter $\lambda > 0$. Then formula (1) of theorem 1 yields

$$E\left[Z(t)Z(t+h)\right] = \lambda E\left[X_1^2\right]\left(\frac{1-e^{-2\delta t}}{2\delta}\right) + \frac{\lambda^2 E^2\left[X_1\right]}{\delta^2}\left(1-e^{-\delta t}\right)\left(1-e^{-\delta(t+h)}\right) .$$

Thus

$$Cov\left[Z(t)Z(t+h)\right] = \lambda E\left[X_1^2\right]\left(\frac{1-e^{-2\delta t}}{2\delta}\right) ,$$

which is independent of h (and then equal to $V\left[Z(t)\right]$) and is almost constant for large t .

Furthermore, if $\rho(t,h)$ is the correlation coefficient between $Z(t)$ and $Z(t+h)$, then

$$\rho(t,h)=\frac{\lambda E\left[X_1^2\right]\left(\dfrac{1-e^{-2\delta t}}{2\delta}\right)}{\left[\lambda E\left[X_1^2\right]\left(\dfrac{1-e^{-2\delta t}}{2\delta}\right)\right]^{1/2}\left[\lambda E\left[X_1^2\right]\left(\dfrac{1-e^{-2\delta(t+h)}}{2\delta}\right)\right]^{1/2}}$$

$$=\left[\frac{1-e^{-2\delta t}}{1-e^{-2\delta(t+h)}}\right]^{1/2} .$$

So $\rho(t,h)\to\left[1-e^{-2\delta t}\right]^{1/2}$ when $h\to\infty$, and $\rho(t,h)$ is almost 0 for a small t and a large h ... as normally expected.

For our discounted compound Poisson process, if the correlation is ‘‘strong enough’’ on the period $[t, t+h]$ then we can eventually use a linear predictor to estimate the value of $Z(t+h)$ from a known value of $Z(t)$.

Hence assume that the equation of the linear predictor is given by

$$L(t,h) = E[Z(t+h)] + \rho(t,h)\left\{\frac{V[Z(t+h)]}{V[Z(t)]}\right\}^{1/2}\{Z(t) - E[Z(t)]\} ,$$

then, for our example, we get

$$L(t,h) = Z(t) + e^{-\delta t}\left\{\frac{\lambda}{\delta}E[X_1](1-e^{-\delta h})\right\} = Z(t) + e^{-\delta t}E[Z(h)] .$$

Joint moments for a stochastic $\delta(x)$

If we now consider a stochastic force of real interest, the preceding method (that use renewal arguments) does not work anymore and, which more is, it is not possible to get recursive formulas for the joint moments.

So we need a more general method that will help us to find explicit formulas for the joint moments of our risk process for a stochastic discount rate. This method will be based essentially on the following lemma that gives the conditional joint distribution of the claims arrival times knowing the number of claims, for any renewal process. This lemma generalizes the well-known similar formulas obtained for the Poisson process by using the order statistic property.

Lemma 2 : Consider an ordinary or a delayed renewal counting process. Then, for $0 = x_0 < x_1 < x_2 < \ldots < x_k \le t$, $i_0 = 0$, $1 \le i_1 < i_2 < \ldots < i_k \le n$ and $1 \le k \le n$, the conditional joint density probability functions of $T_{i_1}, T_{i_2}, \ldots, T_{i_k} \mid N(t) = n$ or $T_{i_1}, T_{i_2}, \ldots, T_{i_k} \mid N_d(t) = n$ are given by :

(1) For the ordinary case :

$$f_{T_{i_1}, T_{i_2}, \ldots, T_{i_k} \mid N(t)}(x_1, x_2, \ldots, x_k \mid n) = \frac{P\left(N\left(t - x_k\right) = n - i_k\right) \prod_{j=1}^{k} f_{T_{i_j - i_{j-1}}}\left(x_j - x_{j-1}\right)}{P(N(t) = n)} .$$

(2) For the delayed case :

$$f_{T_{i_1}, T_{i_2}, \ldots, T_{i_k} \mid N_d(t)}(x_1, x_2, \ldots, x_k \mid n) = \frac{P\left(N_d\left(t - x_k\right) = n - i_k\right) \prod_{j=1}^{k} f_{T_{i_j - i_{j-1}}}\left(x_j - x_{j-1}\right)}{P(N_d(t) = n)} .$$

Proof : We only prove the ordinary renewal case. Thus, we have

$$P\left(T_{i_1} \le x_1, \ldots, T_{i_k} \le x_k \,\middle|\, N(t) = n\right)$$

$$= \frac{P\left(N(t) = n \,\middle|\, T_{i_1} \le x_1, \ldots, T_{i_k} \le x_k\right) P\left(T_{i_1} \le x_1, \ldots, T_{i_k} \le x_k\right)}{P\left(N(t) = n\right)}$$

$$= \frac{\int_0^{x_1}\int_{u_1}^{x_2}\cdots\int_{u_{k-1}}^{x_k} P\left(N(t) = n \,\middle|\, T_{i_1} = u_1, \ldots, T_{i_k} = u_k\right) f_{T_{i_1},\ldots,T_{i_k}}(u_1, \ldots, u_k)\, du_k \ldots du_1}{P\left(N(t) = n\right)}$$

$$= \frac{\int_0^{x_1}\int_{u_1}^{x_2}\cdots\int_{u_{k-1}}^{x_k} P\left(N(t - u_k) = n - i_k\right) \prod_{j=1}^{k} f_{T_{i_j - i_{j-1}}}\left(u_j - u_{j-1}\right) du_k \ldots du_1}{P\left(N(t) = n\right)} \quad .$$

The result follows by taking the appropriate k partial derivatives of the preceding expression. □

Theorem 2 : According to the assumptions of our risk model, and for a stochastic force of real interest, the first three joint moments of $Z(t)$ and $Z(t+h)$ are given, for $t>0$ and $h>0$, by :

(1) $$E\left[Z(t)Z(t+h)\right]=E\left[Z^2(t)\right] + E^2\left[X_1\right]\int_0^t \int_{t-u}^{t+h-u} E\left[D(u)D(u+v)\right]dm(v)\,dm(u) ,$$

where

$$E\left[Z^2(t)\right]=E\left[X_1^2\right]\int_0^t E\left[D^2(u)\right]dm(u) + 2E^2\left[X_1\right]\int_0^t \int_0^{t-u} E\left[D(u)D(u+v)\right]dm(v)\,dm(u) .$$

(2) $$E\left[Z^2(t)Z(t+h)\right]=E\left[Z^3(t)\right]$$

$$+E\left[X_1^2\right]E\left[X_1\right]\int_0^t\int_{t-u}^{t+h-u}E\left[D^2(u)D(u+v)\right]dm(v)dm(u)$$

$$+2E^3\left[X_1\right]\int_0^t\int_0^{t-u}\int_{t-u-v}^{t+h-u-v}E\left[D(u)D(u+v)D(u+v+w)\right]dm(w)dm(v)dm(u)\ ,$$

where

$$E\left[Z^3(t)\right]=E\left[X_1^3\right]\int_0^t E\left[D^3(u)\right]dm(u)$$

$$+3E\left[X_1^2\right]E\left[X_1\right]\int_0^t\int_0^{t-u}E\left[D^2(u)D(u+v)\right]dm(v)dm(u)$$

$$+3E\left[X_1^2\right]E\left[X_1\right]\int_0^t\int_0^{t-u}E\left[D(u)D^2(u+v)\right]dm(v)dm(u)$$

$$+6E^3\left[X_1\right]\int_0^t\int_0^{t-u}\int_0^{t-u-v}E\left[D(u)D(u+v)D(u+v+w)\right]dm(w)dm(v)dm(u)\ .$$

(3) $$E\left[Z(t)Z^2(t+h)\right]=E\left[Z^3(t)\right]$$

$$+3E\left[X_1^2\right]E\left[X_1\right]\int_0^t\int_{t-u}^{t+h-u}E\left[D(u)D^2(u+v)\right]dm(v)dm(u)$$

$$+3E\left[X_1^2\right]E\left[X_1\right]\int_0^t\int_{t-u}^{t+h-u}E\left[D^2(u)D(u+v)\right]dm(u)dm(v)$$

$$+4E\left[X_1^3\right]\int_0^t\int_0^{t-u}\int_{t-u-v}^{t+h-u-v}E\left[D(u)D(u+v)D(u+v+w)\right]dm(w)dm(v)dm(u)$$

$$+2E\left[X_1^3\right]\int_0^t\int_{t-u}^{t+h-u}\int_0^{t+h-u-v}E\left[D(u)D(u+v)D(u+v+w)\right]dm(w)dm(v)dm(u).$$

Proof : We illustrate the main ideas of the proof by solving the first result of our theorem. Thus, let us first obtain an expression for the joint moments generating function, for any integrable function $\delta(x)$ corresponding to a sample path of the force of real interest on the period $[0, t+h]$.

As in lemma1, by conditionning on $N(t)$, $N(t+h)$, $T_1,\dots,T_{N(t+h)}$, we get

$$E\left[e^{xZ(t)+yZ(t+h)}\,\middle|\,\boldsymbol{\delta}(z), z\in[0,t+h]\right]$$
$$=E\left[\prod_{k=1}^{N(t)} M_{X_1}\left((x+y)D(T_k)\right)\prod_{k=N(t)+1}^{N(t+h)} M_{X_1}\left(yD(T_k)\right)\,\middle|\,\boldsymbol{\delta}(z), z\in[0,t+h]\right].$$

An evaluation of the appropriate partial derivatives at $(x,y)=(0,0)$ gives

$$E\left[Z(t)Z(t+h)\,\middle|\,\boldsymbol{\delta}(z), z\in[0,t+h]\right]=E[X^2]E\left[\sum_{k=1}^{N(t)} D^2(T_k)\,\middle|\,\boldsymbol{\delta}(z), z\in[0,t+h]\right]$$
$$+2E^2[X]E\left[\sum_{k=1}^{N(t)-1}\sum_{j=k+1}^{N(t)} D(T_k)D(T_j)\,\middle|\,\boldsymbol{\delta}(z), z\in[0,t+h]\right]$$
$$+E^2[X]E\left[\sum_{k=1}^{N(t)}\sum_{j=N(t)+1}^{N(t+h)} D(T_k)D(T_j)\,\middle|\,\boldsymbol{\delta}(z), z\in[0,t+h]\right].$$

Now, if we condition first on $N(t)$ and thereafter use lemma 2, the first term of the preceding summation gives

$$E\left[\sum_{k=1}^{N(t)} D^2(T_k) \middle| \delta(z), z\in[0,t+h]\right] = E\left[E\left[\sum_{k=1}^{N(t)} D^2(T_k) \middle| N(t), \delta(z), z\in[0,t]\right]\right]$$

$$= \sum_{n=0}^{\infty}\sum_{k=1}^{n}\int_0^t D^2(u) f_{T_k}(u) P\big(N(t-u)=n-k\big)\,du$$

$$= \sum_{k=1}^{\infty}\sum_{n=k}^{\infty}\int_0^t D^2(u) f_{T_k}(u) P\big(N(t-u)=n-k\big)\,du$$

$$= \sum_{k=1}^{\infty}\int_0^t D^2(u)\, dF_{\tau_1}^{*k}(u)$$

$$= \int_0^t D^2(u)\, d\sum_{k=1}^{\infty} F_{\tau_1}^{*k}(u)$$

$$= \int_0^t D^2(u)\, dm(u) \quad .$$

Similarly, it can be proved that

$$E\left[\sum_{k=1}^{N(t)-1}\sum_{j=k+1}^{N(t)} D\left(T_k\right)D\left(T_j\right)\middle|\,\boldsymbol{\delta}(z), z\in\left[0,t+h\right]\right]=\int_0^t\int_0^{t-u} D(u)\,D(u+v)\,dm(v)dm(u)\ ,$$

and

$$E\left[\sum_{k=1}^{N(t)}\sum_{j=N(t)+1}^{N(t+h)} D\left(T_k\right)D\left(T_j\right)\middle|\,\boldsymbol{\delta}(z), z\in\left[0,t+h\right]\right]=\int_0^t\int_{t-u}^{t+h-u} D(u)\,D(u+v)\,dm(v)\,dm(u)$$

Finally, as each of the three preceding integrals are random variables, the result follows by taking the expectation of each one of them, which is equal to the integral of the expectation from a well known theorem of stochastic processes theory. □

Example 2: Let $\{\delta(t), t \geq 0\}$ be an Itô process satisfying the stochastic differential equation of Ho-Lee-Merton

$$d\delta(t) = r dt + \sigma\, dB(t),$$

with constant drift r and constant diffusion coefficient σ, and where $B(t)$ is a standard Brownian motion.

Then we easily obtain

$$E\left[D^2(u)\right] = \exp\left\{-2\delta(0)u - ru^2 + \frac{2}{3}\sigma^2 u^3\right\},$$

and

$$E\left[D(u)D(u+v)\right] = \exp\left\{-\left[\delta(0)(v+2u)\right] - \frac{r}{2}\left[v^2 + 2uv + 2u^2\right] + \frac{\sigma^2}{2}\left[\frac{(v+2u)^3 + v^3}{6}\right]\right\}.$$

If we let $G(t,h)=E\left[Z(t)Z(t+h)\right]$, $\boldsymbol{\tau}_k \sim \exp(\boldsymbol{\lambda}=1)$, $E\left[X_1\right]=1$, $E\left[X_1^2\right]=2$, $\delta(0)=0.03$, $r=0.002$ and $\sigma=0.001$, then the following tables could be obtained from formula (1) of theorem 2

Table 1. $G(t,10)$ --- Ho-Lee-Merton case

t	1	5	10	15	20
$G(t,10)$	10.8372	60.6696	127.4541	188.2064	237.0777
t	30	40	50	60	70
$G(t,10)$	297.3271	322.2795	330.5541	332.8062	333.3136

Table 2. $G(5,h)$ --- Ho-Lee-Merton case

h	5	10	15	20	25
$G(5,h)$	47.1111	60.6696	70.7323	77.8408	82.6212
h	30	35	45	55	65
$G(5,h)$	85.6819	87.5478	89.2301	89.7039	89.8140

Remarks : (1) The first moment of $Z(t)$, for the preceding force of interest, is given by the following expression :

$$E[Z(t)] = E[X_1]\int_0^t E[D(v)]\,dm(v)$$

$$= E[X_1]\int_0^t \exp\left\{-\delta(0)v - r\frac{v^2}{2} + \sigma^2\frac{v^3}{6}\right\}dm(v)\ .$$

(2) For the values of the preceding example, the following table is obtained for $G(t,0) = E\left[Z^2(t)\right]$:

Table 3. $G(t,0)$ --- Ho-Lee-Merton case

t	1	5	10	15	20
$G(t,10)$	2.9098	29.7246	84.4707	145.9729	202.1786
t	30	40	50	60	70
$G(t,10)$	280.0772	315.9861	328.7406	332.3814	333.2318

Regression formulas

Theorem 3 : Let $t>0$, $h>0$, $\Sigma_{t,n}=\left\{N(t)=n,\, T_i=t_i,\, X_i=x_i\ ;\ i=1,\dots,n\right\}$ and $\boldsymbol{\delta}(x)$ be a stochastic force of real interest. Then, according to the assumptions of our risk model, we have the following regression formula :

$$E\left[Z(t+h)\middle|\Sigma_{t,n}\right]=Z(t)$$

$$+E\left[X_1\right]\int_t^{t+h}\left\{E\left[D(x)\right]+\int_0^{t+h-x}E\left[D(x+y)\right]dm(y)\right\}\frac{f_{\tau_1}(x-t_n)}{\overline{F}_{\tau_1}(t-t_n)}dx.$$

Proof : We have,

$$E\left[Z(t+h)\middle|\Sigma_{t,n},\boldsymbol{\delta}(z),z\in[0,t+h]\right]=Z(t)$$

$$+E\left[X_1\right]E\left[\sum_{k=n+1}^{N(t+h)}D(T_k)\middle|\Sigma_{t,n},\boldsymbol{\delta}(z),z\in[0,t+h]\right].$$

By conditionning on T_{n+1} and $N(t+h)$, the last factor yields

$$E\left[\sum_{k=n+1}^{N(t+h)} D(T_k) \middle| \Sigma_{t,n}, \boldsymbol{\delta}(z), z \in [0, t+h]\right] =$$

$$\int_t^{t+h} \left\{ D(x) + \int_0^{t+h-x} D(x+y)\, dm(y) \right\} \frac{f_{\boldsymbol{\tau}_1}(x-t_n)}{\bar{F}_{\boldsymbol{\tau}_1}(t-t_n)} dx \ .$$

As the last integral is a random variable, we apply the expectation as we did previously to get the result. □

Remarks : (1) If we set $\boldsymbol{\delta}(x) = \boldsymbol{\delta}$ ($\Rightarrow D(x) = e^{-\delta x}$) and $\boldsymbol{\tau}_k \sim \exp(\boldsymbol{\lambda})$, then our regression curve corresponds exactly to the linear predictor of example 1.

(2) As that is well-known, the regression curve does not generally correspond to the linear predictor.

Theorem 4 : Let $t > 0$, $h > 0$, $\Sigma_{t,n} = \{N(t) = n,\ T_i = t_i,\ X_i = x_i\ ;\ i = 1,\ldots,n\}$ and $\delta(x)$ be a stochastic force of real interest. Then, according to the assumptions of our risk model, we have the following regression formula :

$$
\begin{aligned}
E\left[Z^2(t+h)\middle|\Sigma_{t,n}\right] &= Z^2(t) \\
&+2Z(t)E\left[X_1\right]\int_t^{t+h}\left\{E\left[D(x)\right]+\int_0^{t+h-x}E\left[D(x+y)\right]dm(y)\right\}\frac{f_{\tau_1}(x-t_n)}{\overline{F}_{\tau_1}(t-t_n)}dx \\
&+E\left[X_1^2\right]\int_t^{t+h}\left\{E\left[D^2(x)\right]+\int_0^{t+h-x}E\left[D^2(x+y)\right]dm(y)\right\}\frac{f_{\tau_1}(x-t_n)}{\overline{F}_{\tau_1}(t-t_n)}dx \\
&+2E^2\left[X_1\right]\int_t^{t+h}\int_0^{t+h-x}\Bigg\{E\left[D(x)D(x+z)\right] \\
&\qquad+\int_0^{t+h-x}E\left[D(x+y)D(x+y+z)\right]dm(z)\Bigg\}dm(y)\frac{f_{\tau_1}(x-t_n)}{\overline{F}_{\tau_1}(t-t_n)}dx\ .
\end{aligned}
$$

Conclusion

We have found recursive formulas for the joint moments of the compound renewal sums with discounted claims, for a constant force of real interest. These formulas have been obtained by using essentially renewal arguments. Covariance and correlation coefficient have been given for the discounted compound Poisson process, thus providing an additional tool for the analysis of our risk process.

We have also found explicit formulas for the first joint moments when the discount factor is stochastic. These formulas have been obtained by giving new expressions for the conditional (joint) probability density function of the claims arrival times knowing their number, for any renewal process, extending some well known techniques using order statistics. A numerical example has also been given, showing the calculability of our formulas. Finally a regression formula has been obtained for our risk process.

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